

PhD Thesis

Optimal Sequential Decisions in Hidden-State Models

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Abstract

This doctoral thesis consists of five research articles on the general topic of optimal decision making under uncertainty in a Bayesian framework. The papers are preceded by three introductory chapters.

Papers I and II are dedicated to the problem of finding an optimal stopping strategy to liquidate an asset with unknown drift. In Paper I, the price is modelled by the classical Black-Scholes model with unknown drift. The first passage time of the posterior mean below a monotone boundary is shown to be optimal. The boundary is characterised as the unique solution to a nonlinear integral equation. Paper II solves the same optimal liquidation problem, but in a more general model with stochastic regime-switching volatility. An optimal liquidation strategy and various structural properties of the problem are determined.

In Paper III, the problem of sequentially testing the sign of the drift of an arithmetic Brownian motion with the 0-1 loss function and a constant cost of observation per unit of time is studied from a Bayesian perspective. Optimal decision strategies for arbitrary prior distributions are determined and investigated. The strategies consist of two monotone stopping boundaries, which we characterise in terms of integral equations.

In Paper IV, the problem of stopping a Brownian bridge with an unknown pinning point to maximise the expected value at the stopping time is studied. Besides a few general properties established, structural properties of an optimal strategy are shown to be sensitive to the prior. A general condition for a one-sided optimal stopping region is provided.

Paper V deals with the problem of detecting a drift change of a Brownian motion under various extensions of the classical Wiener disorder problem. Monotonicity properties of the solution with respect to various model parameters are studied. Also, effects of a possible misspecification of the underlying model are explored.

Keywords: sequential analysis, optimal stopping, optimal liquidation, drift uncertainty, incomplete information, stochastic filtering.

*To decision makers
ignorant of Bayes' rule.*

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I Ekström E., Vaicenavicius J.
Optimal liquidation of an asset under drift uncertainty.
SIAM Journal on Financial Mathematics, vol. 7, no. 1, 2016, 357-381.
- II Vaicenavicius J.
Asset liquidation under drift uncertainty and regime-switching volatility.
arXiv:1701.08579, submitted, 2017.
- III Ekström E., Vaicenavicius J.
Sequential testing of the drift of a Brownian motion.
ESAIM: Probability and Statistics, vol. 19, 2015, 626-648.
- IV Ekström E., Vaicenavicius J.
Optimal stopping of a Brownian bridge with an unknown pinning point.
arXiv:1705.00369, submitted, 2017.
- V Ekström E., Vaicenavicius J.
Wiener disorder detection under disorder magnitude uncertainty.
Manuscript, 2017.

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Also published by the author but not included in this thesis:

- Hambly B., Vaicenavicius J.
The 3/2-model as a stochastic volatility approximation for a large-basket price-weighted index.
International Journal of Theoretical and Applied Finance, vol. 18, 2015, ID 1550041

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1. Introduction

In almost any sphere of human life, we face uncertainty. We do not know for sure whether a raincoat will be of any use when packing up a backpack the night before a hike. Nor we know the exact course of the stock market and so our future pension invested in it. Not to mention that an unusual sense of enervation might be a consequence of a tough week at work, but could also signify an onset of a life-threatening disease.

Amidst unavoidable uncertainties, we have to make decisions and preferably good ones. Should I take a possibly superfluous raincoat? Should a fund manager sell an underperforming stock? Should a doctor after a basic check-up send a fatigued patient for an invasive and expensive test?

Though we cannot avoid making decisions, we often have the flexibility to delay them. This raises an additional question ‘when is it best to make a decision?’. Shall we postpone packing the raincoat to just before leaving, hoping for a more clear-cut weather forecast in the morning? Shall the fund manager hold the unlucky stock for a bit longer for it to bounce back up? Shall the doctor conduct an invasive test only if the patient’s condition does not improve sufficiently fast while resting?

When and what decision it is optimal to make is the general quest of this thesis. Optimal decision-making strategies are found or otherwise investigated in three statistical problems – hypothesis testing, disorder detection, and sampling termination – as well as in a fundamental financial problem of when to sell an asset. To answer when and what to decision to take, precise interpretations to the ambiguous everyday notions of uncertainty, information, learning, and optimality are needed. Luckily, all these notions can be made exact using the mathematical language of Probability Theory and Bayesian Statistics which we will use to work out answers to the decision-making problems of the thesis. The problems studied have either a statistical or a financial motivation; we introduce them next.

A common situation in engineering applications is that we want to make a decision about some quantity X which we cannot measure directly. The quantity may possibly vary with time, so we write $X = \{X_t\}_{t \geq 0}$. Instead, we can only observe the quantity of interest obscured by noise, i.e. we observe

$$Z_t = X_t + \text{“noise”}.$$

1.1 Hypothesis testing problem

In hypothesis testing, the quantity X is an unknown constant. There we want to decide quickly and accurately whether X is above or below a given threshold θ . In a more statistical language, we want to decide on one of the two hypotheses

$$H_- : X < \theta, \quad H_+ : X \geq \theta$$

as early and as accurately as possible.

In a classical decision-theoretic formulation, the need to decide soon and accurately is enforced by the following costs. We have to pay

- cost c per unit time of observation,
- penalty w_+ for a wrong decision when H_+ is true,
- penalty w_- for a wrong decision when H_- is true.

The goal is to find a decision rule that tells us when and what to decide so as to pay the smallest total cost on average.

1.2 Disorder detection problem

In other applications, the quantity X is a known constant at the start, representing a ‘good’ state of the system, but later, at some random time Θ , changes to another constant, interpreted as a ‘disorder’ state. We want to detect the disorder state as early as possible after it has occurred, but also we want to avoid raising a false alarm when a system is in a good state.

In a standard formulation, the requirements to detect the disorder state soon and to avoid a false alarm are imposed by the following costs:

- cost c per unit time after the occurrence of a disorder,
- penalty w for a false alarm.

As before, the goal is to find a decision rule telling us when to raise an alarm in order to pay the smallest total cost on average.

1.3 Sampling termination problem

Another seemingly different however mathematically related problem concerns sampling termination. Suppose (x_1, x_2, \dots, x_n) is a large finite population of real numbers whose population mean we do not know. We are allowed to randomly sample without replacement from this population in a sequential manner and stop whenever we wish based on the values drawn. Each number drawn represents gain (loss if negative) and the cumulative sum process represents the total gain from the samples drawn thus far. The goal is to find a stopping rule telling when to stop sampling (based on the values drawn) so that the total gain from sampling is maximal on average.

1.4 Optimal liquidation problem

In financial markets, prices of goods fluctuate over time, so everyone possessing an asset and willing to exchange it for money faces the question ‘When shall I sell the asset to benefit the most from the sale?’. However, to a large extent the future price movements of financial assets are unpredictable, so the aforementioned question does not have an answer. It is simply not a right question to ask. A more sensible version of the same question is the following: ‘When shall I sell the asset to benefit the most from the sale *on average*?’ In the thesis, this refined question is studied extensively in situations where the asset needs to be sold before a fixed time in the future.

2. Preliminaries

Before embarking on a search for optimal decision rules, appearing in later parts of the thesis, we need answers to the following crucial questions.

- How to model uncertainty?
- How to model acquisition of knowledge from the available information, or more simply put, learning?
- How to compare decision rules and what does it mean to be optimal?
- What methods are already available for finding optimal decision rules?

Thankfully, we do not have to start from scratch but can rely on mathematical frameworks and tools developed over the last century within the fields of stochastic analysis, stochastic filtering, statistical decision theory, and optimal stopping theory. A detailed coverage of these topics spans hundreds of pages and cannot be included in a thesis like this. Instead, the objective of the rest of this chapter is to convey the most important concepts and ideas. In addition, directions to more in-depth sources are provided for the interested reader.

2.1 Uncertainty as a probability distribution

Uncertainty is not the same as complete absence of knowledge. We might know a lot about a system of interest (e.g. the general physical laws it obeys), but still not know its exact current state. This happens when the system is affected by factors that we cannot control or understand sufficiently well. Such influences are called *random*, and the affected system itself *stochastic*. In situations where there are many small random factors affecting the system, we simply say that the system is affected by *noise*.

In the Bayesian paradigm of Statistics, our uncertainty or lack of knowledge about something is represented (modelled) by a probability distribution. The intuitive interpretation is that the number of times we believe more in outcome A than in outcome B , the number of times the probability density/mass is greater at outcome A than at outcome B . In the thesis, we will adopt this *Bayesian paradigm*. It is worth noting that probability distributions can be defined not only on discrete sets or real numbers (as typically encountered in basic courses) but also on many complicated mathematical structures such as the space of continuous functions. This makes the Bayesian paradigm a very general and widely applicable principle. For a more in-depth discussion and motivation of the Bayesian paradigm the reader is referred to [7, Chapter 1] or [5, Chapter 3].

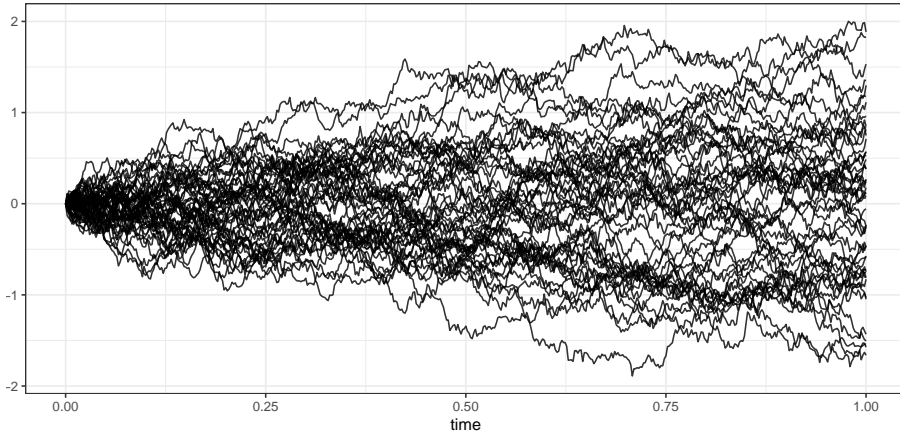


Figure 2.1. 50 realisations of the Brownian motion B , a fundamental noise process.

2.2 Stochastic differential equations: models for noisy systems

One of the goals of science is to make and maintain our knowledge – including our knowledge about uncertainty – systematic in the form of mathematical models. Many physical systems with negligible random influences are successfully modelled by differential equations, however, for stochastic systems, equations incorporating the random perturbations of the system are required. Such equations have been developed and are called *stochastic differential equations*. In this thesis, we will make heavy use of stochastic differential equations having the integral form

$$X_t = \underbrace{\int_0^t b(u, X_u) du}_{\text{usual differential equation part}} + \underbrace{\int_0^t \sigma(u, X_u) dB_u}_{\text{additional noise perturbations}}, \quad t \geq 0. \quad (2.1)$$

The second term on the right-hand side of (2.1) is not a regular integral, but a special stochastic integral, known as *the Itô integral*. It is taken with respect to a canonical noise process, called *the Brownian motion* (see Figure 2.1). Recall that a solution to a classical differential equation is a deterministic function of time, i.e we know the exact state of the system at any time. In contrast, a solution to an stochastic differential equation is a family of probability distributions indexed by time. Each probability distribution expresses our uncertainty about the state of the system at a particular time.

A whole theory of SDEs has been developed since Itô's definition of a stochastic integral in mid-1940's [8]. The field is also known as the *Itô Calculus*, or *Stochastic Calculus*. Appreciating the central role of differential equations in science over the last few centuries, it is no surprise that stochastic

differential equations have been applied in many scientific disciplines, including Physics, Engineering, Finance, and Genetics.

A vast literature exists on the topic of stochastic differential equations, however, here we can only give a few pointers. Among many texts written on the subject, standard references on stochastic differential equations driven by a Brownian motion are [9] and [18]. Besides the Brownian motion, Ito's definition of the stochastic integral extends to a large class of processes, called *semimartingales*. A comprehensive treatment of stochastic calculus with respect to general semimartingales can be found in [3] or [17]. For diverse applications of SDEs, see [14]; for financial applications, see an established monograph [10].

2.3 Stochastic filtering: learning about the hidden state

As discussed before, in this mathematical framework, knowing means 'to know a model' and uncertainty is expressed by a probability distribution. The intuitive notion of 'learning' is thus a process of gradual refinement of the model (initially incorporating much uncertainty) by incorporating newly arriving information. A technical term for such 'learning' in this framework is *stochastic filtering*, often shortened to just *filtering*.

For learning, information is needed, so let us first discuss how information and gradual accumulation of it can be modelled mathematically. In probability theory, information is modelled by a set, called σ -*algebra*, representing all the questions that can be given a definite answer. Formally, σ -algebra is a set of subsets of the sample space Ω that contains the empty set and is closed under countable set operations. Let \mathcal{F}_t denote the σ -algebra corresponding to the information available at time t . Then a gradual accumulation of information can be modelled by the sequence $\{\mathcal{F}_t\}_{t \geq 0}$. It satisfies the property that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$. In general, such an increasing sequence of σ -algebras modelling gradual accumulation of information is called a *filtration*.

We can now overview a common setup of filtering. To illustrate the key concepts without overburdening notation, we will confine ourselves to a one-dimensional case. We are interested in a quantity X , known as *signal* or *hidden state*, but we cannot observe it directly. Instead, we can only observe a process Y whose value depends both on the signal process as well as noise in the following way:

$$Y_t = Y_0 + \int_0^t h(X_u) du + W_t. \quad (2.2)$$

Here h is a real-valued function, called the *sensor function* and W is a Brownian motion independent of X . Casually put, the main question of the filtering theory is 'what is the most we can know about X_t given observations of Y up to time t ?'. Reformulated in the language of probability theory, the question

becomes ‘what is the conditional distribution of X_t given the σ -algebra \mathcal{F}_t^Y ?’ . As described above, \mathcal{F}_t^Y represents the information gathered from continuous observation of Y up to time t .

To find the conditional distribution, a purpose-built incarnation of Bayes’ rule, known as the *Kallianpur-Striebel formula*, can be used. It says that for any measurable subset E of the state-space of X ,

$$\mathbb{P}(X_t \in E | \mathcal{F}_t^Y) = \frac{\int_{x_t \in E} \psi_t(x, Y) dF(x)}{\int_{\mathcal{D}} \psi_t(x, Y) dF(x)},$$

where

$$\psi_t(x, Y) = \exp \left(\int_0^t h(x_u) dY_u - \frac{1}{2} \int_0^t h(x_u)^2 du \right)$$

and F is a probability measure on the set of possible trajectories \mathcal{D} of the process X .

Suppose X is a *Markov process*, a stochastic process whose future evolution depends only on its current state. Then the conditional distribution satisfies the *Kushner-Stratonovich equation*

$$\begin{aligned} \mathbb{E}[\varphi(X_t) | \mathcal{F}_t^Y] &= \mathbb{E}[\varphi(X_t)] + \int_0^t \mathbb{E}[(A(\varphi))(X_u) | \mathcal{F}_u^Y] du \\ &+ \int_0^t \mathbb{E}[\varphi(X_u)h(X_u) | \mathcal{F}_u^Y] - \mathbb{E}[\varphi(X_u) | \mathcal{F}_u^Y]\mathbb{E}[h(X_u) | \mathcal{F}_u^Y] dI_u \end{aligned}$$

for any function φ from an appropriate set of test functions. In the above, A denotes the generator of the Markov process X (the generator is a special operator characterising a Markov process, see e.g. [6]). The process $I_u = Y_u - \int_0^u \mathbb{E}[h(X_s) | \mathcal{F}_s^Y] ds$ and is known as the *innovation process*.

The general filtering theory in continuous time is mathematically demanding, making use of abstract measure theory, probability theory, as well as real and functional analysis. For an accessible contemporary textbook on filtering in continuous time, see [1]. The classical reference in the field is the two volume book [12], [13]. The general theory of filtering, recent developments, and many applications can be found in the monograph [4].

2.4 Decision strategies and the definition of ‘optimal’

Recall that the quest of the thesis is to find out when and what decision it is optimal to make. In other words, we want to find a pair (τ, d) , where τ is a rule telling us when to stop waiting and d is a rule telling what decision to make at the time we have stopped. We will call the pair (τ, d) a *decision strategy*.

Having an intuitive understanding of what a decision strategy is, let us look into a mathematical definition of this concept within our framework. It is an

obvious law of nature that we cannot peep into the future. Hence any realistic decision strategy (τ, d) can only use the past and not future information. This is encapsulated in the following two requirements. First, the stopping rule τ is a special random variable, known as the stopping time. A random variable $\tau : \Omega \rightarrow [0, \infty]$ is called a *stopping time* (with respect to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ representing available information) if $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. Second, the decision rule d is required to be a \mathcal{F}_τ -measurable random variable with values in some set \mathcal{S} corresponding to the allowed decisions. In the decision problems studied in the thesis, the only source of new information will be the observation of the process Y in (2.2), i.e. the accumulated information filtration $\{\mathcal{F}_t\}_{t \geq 0} = \{\mathcal{F}_t^Y\}_{t \geq 0}$.

Before talking about ranking decision rules under uncertainty, we need to answer how to rank decision rules in situations without any uncertainty. In the Bayesian framework, all the modelling we do happens on a single probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence we can interpret each $\omega \in \Omega$ as a one out of myriads of possible realisations of the world within our model. Intuitively, if we know ω , we know everything that is going to happen within our model, there is no uncertainty left. Suppose we know $\omega \in \Omega$ and assume that the loss (typically interpreted as cost) incurred by a decision strategy (τ, d) can be expressed as a real number $L(\omega, \tau, d)$ (a negative real would correspond to profit). Then the real-valued function $L(\omega) : (\tau, d) \mapsto L(\omega, \tau, d)$, which we may call the *loss function in the known realisation ω* , measures and so ranks the performance of any (τ, d) . We will refer to the function L as the *loss function*.

In the presence of uncertainty, a standard approach to gauge the performance of a decision strategy (τ, d) is by its *expected loss* $\mathbb{E}[L(\tau, d)]$. This allows us to give a rigorous definition to ‘optimal’ in the context of decision strategies. A decision strategy (τ^*, d^*) is said to be *optimal* if

$$\mathbb{E}[L(\tau^*, d^*)] = \inf_{(\tau, d)} \mathbb{E}[L(\tau, d)], \quad (2.3)$$

where the infimum is taken over possible decision strategies.

In the majority of problems studied in the thesis, stopping also automatically incorporates the decision. In other words, there is only one element in the set \mathcal{S} of allowed decisions. Hence, in those settings, the identity (2.3) reduces to

$$\mathbb{E}[L(\tau^*)] = \inf_{\tau} \mathbb{E}[L(\tau)]. \quad (2.4)$$

We will call such a stopping time τ^* *optimal*. For the curious, a conventional account of statistical decision theory in a Bayesian context can be found in [2].

2.5 Optimal stopping theory

Often a useful approach to solving a decision problem of the form (2.3) or (2.4) is to transform it into an optimal stopping problem for a stochastic pro-

cess without hidden states (we will say that such an optimal stopping problem is *of the standard form*). Though the transformation does not make a difficult problem easy, the advantage is that there are some general techniques developed for studying optimal stopping problems of the standard form. One such technique is known as the *martingale approach* and can be applied to large classes of processes. However, the downside of the method is that actionable strategies are difficult to extract. Another, called the *Markovian approach*, can be applied when the underlying process is a Markov process. For diffusion processes, the method provides a way to interpret the optimal stopping problem as a free-boundary partial differential equation. This link to partial differential equations is exploited frequently in the thesis.

Although a subset stochastic control, by the end of the twentieth century, the optimal stopping theory had matured into a separate research area. The developments in optimal stopping theory have been often driven by applied problems from engineering, statistics, economics, and finance. A monograph [15] dedicated to optimal stopping contains the general theory, solution methods, as well as many applications. For connections to variational inequalities and reflected backward stochastic differential equations, see [16]. In addition, many other useful results in optimal stopping can be found in the books [10], [11], and [14].

3. Contributions

The thesis contributes to greater knowledge and understanding in two different respects:

- (i) by solving or otherwise investigating important sequential decision problems from applications,
- (ii) by advancing along the way mathematical techniques relevant for tackling other similar problems.

This chapter contains short summaries of the articles constituting the thesis. For more details, including the context, precise problem formulations, full statements of the results, and proofs, please refer to the original papers that follow.

3.1 Optimal liquidation under drift uncertainty – Paper I

In this first paper, we study a problem of finding an optimal stopping strategy to liquidate an asset with unknown drift before a fixed future time $T > 0$. The asset price is modelled by the classical Black-Scholes model

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{3.1}$$

and the objective is to find a stopping time τ^* such that

$$\mathbb{E}[S_{\tau^*}] = \sup_{\tau} \mathbb{E}[S_{\tau}], \tag{3.2}$$

where the supremum is taken over stopping times with respect to the filtration generated by S . The known drift assumption in the Black-Scholes model is recognised as unrealistic (e.g. see [19, Section 4.2 on p. 144]). Taking a Bayesian approach, we model the uncertainty, or initial beliefs of an individual, about the true drift α by a probability distribution.

In this paper, the optimal liquidation problem (3.2) is solved under an arbitrary prior distribution for the drift, i.e. for any possible uncertainty or set of beliefs about the drift. The first time the posterior mean of the drift passes below a specific non-decreasing curve is shown to be optimal. The stopping boundary is characterised as the unique solution of a particular integral equation. In addition, we derive some sufficient conditions for monotonicity in the volatility σ as well as the prior distribution. Finally, the effects of parameter uncertainty and filtering on the expected optimal selling payoff are illustrated via numerical experiments in the case of the normal prior.

3.2 Liquidation under drift uncertainty and regime-switching volatility – Paper II

In the Black-Scholes model used in Paper III, the volatility σ is treated as constant. However, it is widely acknowledged that more realistic models should incorporate stochastic volatility, which is observed empirically. To address this imperfection, Paper II solves the same optimal liquidation problem under drift uncertainty, but in the presence of stochastic volatility.

The undertaking can be briefly summarised as follows. Uncertainty about the drift is represented by an arbitrary probability distribution. Stochastic volatility is modelled by an m -state Markov chain, often referred to as *regime-switching volatility*. Using filtering theory, an equivalent reformulation of the original problem as a four-dimensional optimal stopping problem is found and then analysed by constructing approximating sequences of three-dimensional optimal stopping problems. An optimal liquidation strategy and various structural properties of the problem are determined. Analysis of the two-point prior case is presented in detail. Building on it, an outline of the extension to the general prior case is given.

In a broader mathematical context, the investigated selling problem appears to be the first optimal stopping problem with parameter uncertainty and stochastic volatility to be studied in the literature. Thus it is plausible that some of the ideas presented in the article will find use in other optimal stopping problems of the same type.

3.3 Testing the drift of a Brownian motion – Paper III

Paper III investigates a Bayesian statistics problem of sequentially testing the sign of the drift of an arithmetic Brownian motion with the 0-1 loss function and a constant cost of observation per unit of time. Though the problem is classical, an actionable optimal decision strategy has been known only for a two-point prior (see [20]). In the paper, we determine and investigate an optimal decision strategy for an arbitrary prior distribution. An optimal decision strategy is expressed as the first entry time of a particular conditional probability process to a special stopping set. The stopping set is determined by two monotone boundaries, which we characterise in terms of integral equations.

Let us briefly summarise our enterprise. First, the sequential decision problem is reformulated as an optimal stopping problem with the current conditional probability that the drift is non-negative as the underlying process. Using filtering theory as well as probabilistic techniques, the conditional probability process is shown to be a martingale diffusion with the dispersion function non-increasing in time. The derived properties of the underlying process enable us to show that the Markovian value function is convex, continuous, and increasing in time. This leads to the existence of two monotone stopping

boundaries, the top one non-increasing, the lower one non-decreasing, which are also proved to be continuous. The boundaries are characterised completely in the finite-horizon case as the unique continuous solution to a pair of integral equations. In the infinite-horizon case, the boundaries are shown to solve another pair of integral equations, moreover, a convergent approximation scheme for the boundaries is provided. Regarding dependence of the strategy on the prior, we uncover how the long-term asymptotic behaviour of the boundaries depends on the support of the prior.

3.4 Stopping a Brownian bridge with an unknown pinning point – Paper IV

A Brownian motion on a finite time interval conditioned to end at a particular value (known as the *pinning point*) at the terminal time is called the *Brownian bridge*. In this article, the problem of stopping a Brownian bridge with unknown pinning point to maximise the expected value at the stopping time is studied. The problem has a statistical interpretation of maximising expected total gain from sequential sampling without replacement from a large finite population when we do not know the exact mean of the population.

A Bayesian formulation of the problem is studied in which the initial beliefs (or knowledge) about the pinning point are described by an arbitrary prior distribution. Structural properties of an optimal stopping strategy are shown to crucially depend on the prior. Still, a few general properties (e.g. continuity of the Markovian value function) are established to hold under an arbitrary prior. Also, we provide a general condition for a one-sided stopping region. More detailed analysis is conducted in the cases of the two-point and the mixed Gaussian priors, revealing a rich structure present in the problem.

3.5 Wiener disorder detection – Paper V

In this final paper, we study the problem of detecting a drift change of a Brownian motion under various extensions of the classical case [20]. In our general set-up, the signal process X is modelled by

$$X_t = B^0 \mathbb{1}_{\{\Theta=0\}} + B^1 \mathbb{1}_{\{0 < \Theta \leq t\}}.$$

Here Θ is a random variable representing the disorder occurrence time. The quantities B^0, B^1 are random variables expressing the magnitude of a disorder. The signal X is not observable directly, instead, we can continuously observe

$$Y_t = \int_0^t X_u du + \int_0^t \sigma(u) dW_u.$$

The objective is to find a disorder detection strategy τ minimising the expected loss

$$\mathbb{E} \left[\mathbb{1}_{\{\tau < \Theta\}} + \int_{\Theta}^{\tau} c_u \, du \right],$$

where $\mathbb{1}_{\{\tau < \Theta\}}$ is the penalty for a false alarm, and c_u is an instantaneous cost of observation per unit time.

In this setting, we study monotonicity and robustness properties of the detection problem. We prove that the minimal expected loss value increases in σ and c . In contrast, stretching out the prior by scaling is shown to decrease the value. Also, we provide a comparison result in terms of the value of the classical time-homogeneous case. Lastly, we explore effects of misspecification of the underlying model.

4. Summary in Swedish

Svensk sammanfattning av avhandlingen

Optimal sequential decisions in hidden-state models

(på sv. ‘Optimala sekventiella beslut i dolda tillståndsmodeller’)

Denna avhandling innefattar fem forskningsartiklar om optimalt beslutsfattande under osäkerhet. Avhandlingen inleds med tre kapitel som motiverar frågorna, ger en översikt över nödvändiga matematiska förkunskaper och sammanfattar avhandlingens forskningsresultat.

I Artikel I söks en optimal stoppstrategi för att bestämma när en tillgång vars pris har okänd drift bör säljas. Priset modelleras med den klassiska Black-Scholesmodellen där driften ses som okänd. Genom att använda ett bayesiansk ramverk modelleras osäkerheten om driften med en godtycklig sannolikhetsfördelning. En optimal tid att sälja tillgången ges då av första tidpunkten som posteriorimedelvärdet underskrider en viss monoton gräns. Gränsen ges som lösningen till en ickelinjär integralekvation. Slutligen studeras monotonicitetsegenskaper med avseende på priorifördelningen och tillgångens volatilitet.

I Artikel II löses samma problem som i Artikel I i det mer generella fallet med stokastisk volatilitet. En optimal strategi härleds och diverse strukturella egenskaper fastställs. Fallet med en tvåpunktspriorifördelning presenteras i detalj. Baserat på detta ges även en skiss av fallet med en generell priorifördelning.

I Artikel III betraktas ett sekventiellt hypotesprövningsproblem där man är intresserad av tecknet på driften hos en aritmetisk Brownsk rörelse. Optimala beslutsstrategier hittas för 0-1-förlustfunktion med fast observationskostnad per tidsenhet och en godtycklig priorifördelning. Dessa strategier ges av första tidpunkten vid vilken en viss betingad sannolikhetsprocess når ett specificerat stoppområde. Områdets rand ges av två monotona gränser vilka beskrivs i termer av integralekvationer.

I Artikel IV undersöks ett stoppproblem där en Brownsk brygga med okänd slutvärde ska stoppas så att väntevärdet vid stopptiden maximeras. Det visas att optimala strategiers strukturella egenskaper är känsliga för val av priorifördelning. Trots detta fastställs ett antal generella egenskaper oberoende av priorifördelning, exempelvis den Markovska värdefunktionens kontinuitet. Dessutom ges ett generellt villkor för att ha en ensidig stoppregion. En detaljerad analys utförs i fallen där priorifördelningen är en tvåpunktsfördelning eller en blandning av normalfördelningar.

Artikel V behandlar problemet att upptäcka förändring i driften av en Brownsk rörelse under diverse utvidgningar av det klassiska Wiener störningsproblemet. Mer specifikt studeras monotonicitetsegenskaper med avseende på olika modellparametrar i fallet där störningen har slumpmässig storlek. även robusthetsegenskaper undersöks, i betydelsen av påverkan av en felaktigt specificerad modell.

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